11. Theory of the Astronomical Tide

11.1 Theoretical Discussion on the Astronomical Tide

The ebb and flood of the astronomical tide are common sights for a person living in a coastal area. The difference in the sea water levels at ebb and flood are 1 m～1.5 m on the Japanese coasts facing the Pacific Ocean. The difference in the sea level due to the tide in Ariake Bay on the west coast of the Kyushu Island is especially large, and its amplitude sometimes exceeds 4 meters. On the other hand, tides are hardly observed on the coasts facing the Sea of Japan.

The difference in the sea level due to astronomical tide on the western coast of the Korean Peninsula is considerably larger than that in Ariake Bay, and sometimes its amplitude exceeds 8 meters. A double-locking gate system has been arranged, and the open sea and water level inside the international trading port of Incheon, which is 40 km west of Seoul, are thus adjusted.

The water level at the point where two big water areas connect in the form of a narrow strait differs considerably sometimes due the difference of the tidal oscillation systems between them. A very rapid sea water current is observed in such straits. The famous Naruto Channel, a channel between Shikoku Island and Awaji Island, is located at the contact point of the water from the Pacific Ocean and the Seto Inland Sea each of which creates independent oscillation systems; this causes severe tidal currents called "Uzushio," i.e., "whirlpool' of Naruto Channel."

Indonesia consists entirely of a chain of archipelagos, and is located between the Pacific and the Indian Oceans; many strong tidal current appears along its coasts.

A tidal phenomenon itself is explainable by classical principles. Since the flow is not completely symmetrical at the ebb and flow, there is "residual current." This plays an active role in sea water mixtures in inner bays such as in Osaka and Tokyo bays.

Furthermore, the astronomical tide is a phenomenon that slows the rotation of the earth. It is well known that the length of one day has become shorter today than what it was in the Paleozoic and the Mesozoic periods. Atomic clocks give precise times, and the delay in the rotation of the earth has been detected directly.

The lunar revolution period influences the rotation; further, as the total rotation momentum of the Earth-Moon system has a fixed value, the rotational speed of the earth decreases.

11.2 Let us consider the cause of tides intuitively

+It is common sense that lunar gravitation is the main cause of astronomical tides. This is well known from ancient days that the times of the ebbs and floods are delayed by approximately 50 minutes every day, in accordance with the daily moonrise time.

[Question] The anecdote of a person living by the sea and knowing the time of ebb and flood of the astronomical tide well is recorded in "Kojiki". Where is this from?

Hint : "Seto of Hayasui"

The ebb and flood usually happen twice every day. This is because the seawater swells at the side in the opposition direction with the moon (Figure 1). This is correct, but is it easily understandable intuitively? Or, Can school children or junior high school students understand this? If you cannot explain this to a junior high school student, it means that you do not understand it yourself.

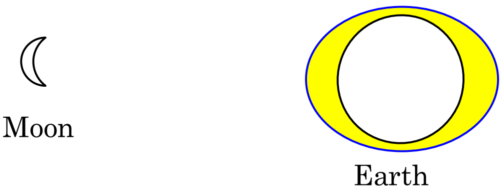


Fig.1 Sea water displacement due to astronomical tide

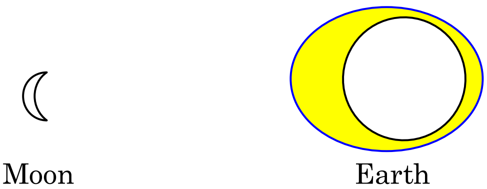


Fig.2 If we only consider that tide is induced by the gravitation of the Moon・・・

Tide is induced by the gravitation of the Moon. Well, then why does the scenario in Fig. 2 not happen? Pachinko balls are placed in all directions with the spring on a wooden disk fixed on a wall and a magnet is placed close to it.

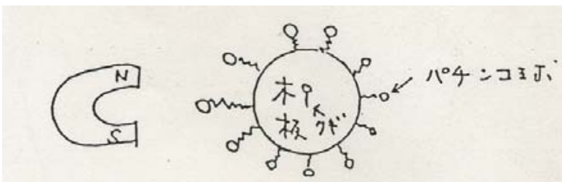


Figure 3 A magnet is placed close to the wooden disk on which pachinko balls are kept in all directions.

We suppose that there is a girl having long hair with a boy who is holding one hand; they are twirling. What will happen to the girl's hair here?

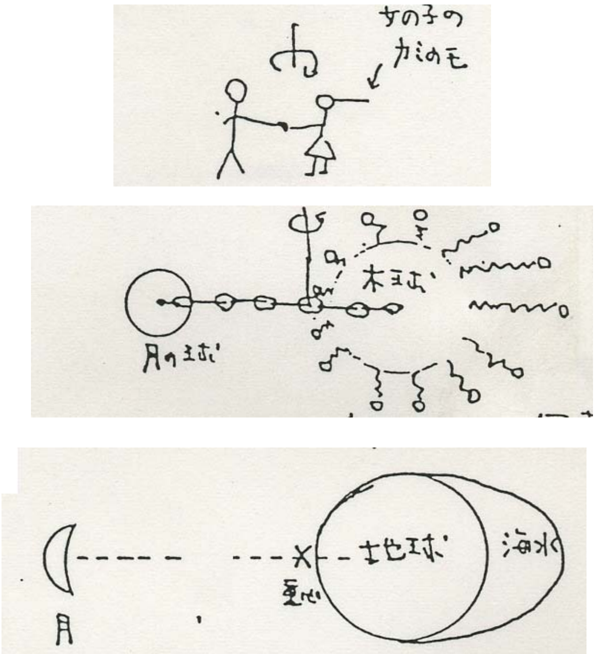


Figure 4. What happens to the hair of a twirling girl? What happens to the pachinko balls attached to the surface of a larger wooden ball tied with the smaller ball by a chain when they are flung?

Well, we can judge that the explanations in Figures 3 and 4 are partly correct. There is neither a nail as shown in Fig. 3 nor a chain as shown in Fig. 4 in the actual Earth-Moon system.

The moon and the Earth are rotating around a common gravitational center. At the center of the Earth, the centrifugal force due to the rotation and the gravitational force of the moon are balanced. However, at a point nearest to the moon, the gravity of the moon exceeds the centrifugal force. (Fig. 5)

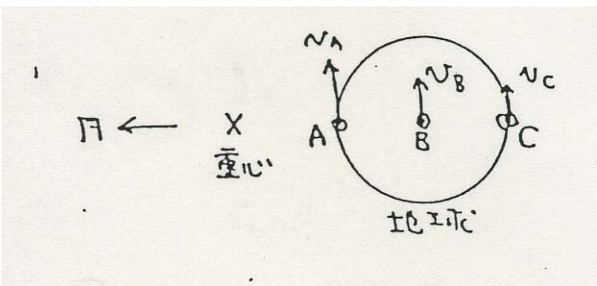


Fig. 5 At the center of the Earth, the centrifugal force due to the rotational motion and the gravitational force due to the moon are balanced. However, at the point A nearest to the moon, the

gravity of the moon exceeds the centrifugal force

Can you understand the fact that the centrifugal force has the same value at any point on Earth? For example, you move a 10 yen coin on the table by pushing it with your finger. The motion of the point P and that of the center of the coin are the same if we move the coin without imparting rotational motion. Therefore, the centrifugal force is the same at any point on the coin.

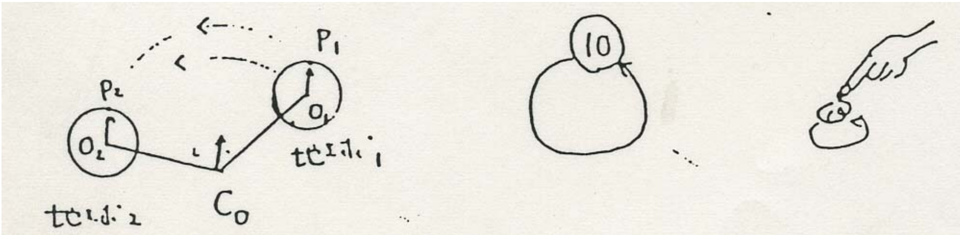


Fig. 6 Centrifugal force has the same value at any point on Earth

11.3 Theoretical Discussions of the Astronomical Tide

Now, let us expand the theoretical discussion.

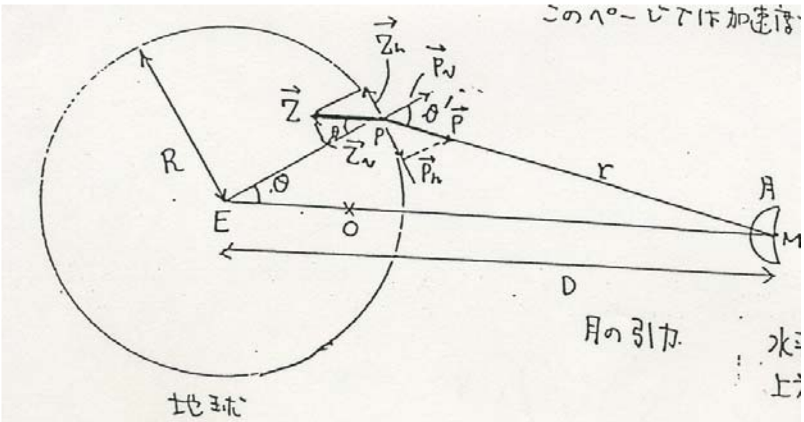


Fig. 7 The Earth-Moon System

We assume points E, M, and O as the centers of the Earth and the Moon and the common gravitational center. Further. we assume P to be the observation point on the Earth's surface, and its latitude (with respect to the Moon) is θ. The radius of the Earth is R (km), and the distance between the Earth and the Moon is D (km). The zenith angle of the moon is assumed to be θ', and the distance between the point P and the center of the moon is r. We put the gravitational force of the moon at P as $\vec{P}$ and the centrifugal force as $\vec{Z}$. Their absolute values are given by the following:

$$\left|\vec{P}\right| = G\frac{M}{r^2} \qquad (1)$$

$$\left|\vec{Z}\right| = \left|EO\right| \times \omega^2 = G\frac{M}{D^2} \tag{2}$$

[Question] We can recognize that (1) is obviously right; however, (2) is not always obvious. Explain the reason why is (2) satisfied?

Hint: See Fig. 6

We write the vector forces by using the horizontal and the vertical components as the follows:

The gravitational force of the moon:

the horizontal component; $\vec{P}_h = G\frac{M}{r^2}\sin\theta'$ (3)

the vertical component; $\vec{P}_v = G\frac{M}{r^2}\cos\theta'$ (4)

and the centrifugal force are written in the same manner

the horizontal component : $\vec{Z}_h = G\frac{M}{D^2}\sin\theta$ (5)

the vertical component : $\vec{Z}_v = G\frac{M}{D^2}\cos\theta$ (6)

The "tidal force" is given by their difference ($\vec{F} = \vec{P} - \vec{Z}$), that is,

the horizontal component : $\vec{F}_h = GM\left(\frac{\sin\theta'}{r^2} - \frac{\sin\theta}{D^2}\right)$ (7)

the vertical component : $\vec{F}_v = GM\left(\frac{\cos\theta'}{r^2} - \frac{\cos\theta}{D^2}\right)$ (8)

Thus we mathematically formulated the tidal force components, in which we have two inconvenient valuables: r and θ'. We eliminate these valuables by using the following geometrical relationships.

$$\sin\theta' = \frac{D}{r}\sin\theta \tag{9}$$

$$\cos\theta' = \frac{D\cos\theta - R}{r} \tag{10}$$

and

$$r^2 = D^2 + R^2 - 2RD\cos\theta \tag{11}$$

By using these relationships, we write both the components of the tidal force as R and θ. Equations (7) and (8) become

$$\vec{F}_h = G\frac{M}{D^2}\left\{\frac{\sin\theta}{\left(1-2(R/D)\cos\theta+R^2/D^2\right)^{3/2}}-\sin\theta\right\} \qquad (12)$$

and

$$\vec{F}_v = G\frac{M}{D^2}\left\{\frac{\cos\theta - R/D}{\left(1-2(R/D)\cos\theta+R^2/D^2\right)^{3/2}}-\cos\theta\right\} \qquad (13)$$

Since R/D is a small number (= 0.017 for the Earth-Moon system), (12) and (13) can be expanded into the following style:

$$\left(1-2(R/D)\cos\theta+R^2/D^2\right)^{-3/2} = 1+3(R/D)\cos\theta+\frac{3}{2}(R/D)^2\left(5\cos^2\theta-1\right)+\frac{5}{2}(R/D)^3\left(7\cos^3\theta-3\cos\theta\right)$$

$$+\ldots\ldots \qquad (14)$$

Then (12) and (13) become

$$\vec{F}_h = G\frac{MR}{D^3}\sin 2\theta+\frac{3}{2}G\frac{MR^2}{D^4}\left(5\cos^2\theta-1\right)\sin\theta+\ldots \qquad (15)$$

$$\vec{F}_v = G\frac{MR}{D^3}\left(3\cos^2\theta-1\right)+\frac{3}{2}G\frac{MR^2}{D^4}\left(5\cos^3\theta-3\cos\theta\right)+\ldots \qquad (16)$$

Ⅲ-rd order　　　Ⅳ-th order

The main part of the tidal force is the term Ⅲ, and the term Ⅳ is smaller by one order.

[Question] We assume the mass and the radius of the Earth to be 1; then, those of the moon are 1/80 and ¼, respectively. The distance between the Earth and the Moon is 60 times that of the radius of the Earth. How many times is the tidal force on the surface of the Moon in comparison to that of the Earth?

Now we give R, D, and M their actual value, and we express the force (acceleration) by the unit of g $(\cong G\frac{M_\oplus}{R^2} = 9.8m/\sec^2)$, the acceleration due to gravity on Earth.

For the Moon

$$\vec{F}_h = 8.4\times10^{-8}\sin 2\theta \times g \qquad (15')$$

$$\vec{F}_h = 5.6\times10^{-8}\left(3\cos^2\theta-1\right)\times g \qquad (16')$$

The tidal forces due to the sun is 0.46 times that of the moon and the coefficients in (15-1) and (16-a) are 3.9 × 10^{-8} and 2.6 × 10^{-8}、respectively.

[Problem] A man weighing 100 kg is on a weighing machine, and at that time, the moon is rising. What will the machine show when the moon reaches the top of the sky (the zenith)?

11.4 The tidal force potential

We can introduce a "tidal force potential" Ω for the tidal force because both gravitational and centrifugal forces are the forces of "conservation."

The tidal force potential Ω is given by the following:

$$\Omega = \frac{1}{2}\frac{GM}{D^3}R^2\left(3\cos^2\theta - 1\right) + \frac{1}{2}\frac{GMR^3}{D^4}\left(5\cos^3\theta - 3\cos\theta\right) \qquad (17)$$

III-rd order term IV-th order term

We can calculate both the components of the tidal force by

$$\vec{F}_h = \frac{\partial\Omega}{\partial r}、\text{ and } \quad \vec{F}_v = \frac{\partial\Omega}{r\partial\theta} \qquad (18)$$

[Problem] What is the definition of "Conservational Forces"?

[Problem] Prove that the acceleration field induced by the centrifugal force is conservational and has a potential.

[Problem] Derive (15) and (16) by using (17) and (18).

11.5 Static tidal theory

If the earth does not rotate, and the surface of the Earth is covered by an ocean of uniform depth, then the sea surface can be theoretically assumed to be the shape of iso-potential plane given by (17).

The depth at the latitude θ is given by the following formula.

$$h(\theta) = \frac{1}{2}\frac{MR^4}{M_\oplus D^3}\left(3\cos^2\theta - 1\right) \qquad (19)$$

where M is the mass of the moon and $M_\oplus$ is that of the Earth.

The tidal difference is 78 cm (during New Moon and Full Moon) and 29 cm(during half moon)

From (19), we can see that the vertical component of the tidal force will be zero at $\cos\theta = \pm\sqrt{1/3}$, that is, θ=55 and 125°.

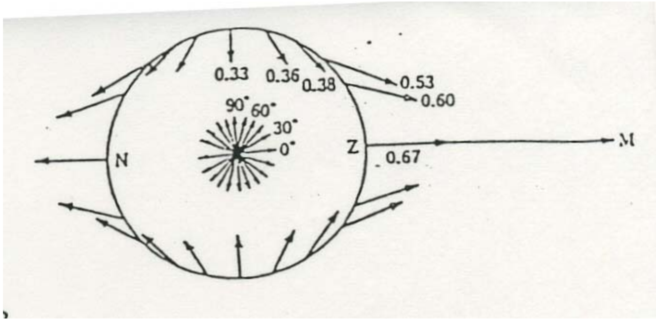


Fig 8 Tide force vector

11.6 Uneven ebb-flood and daily ebb and flood

Let us see the tidal curve at an actual port (Fig. 8)

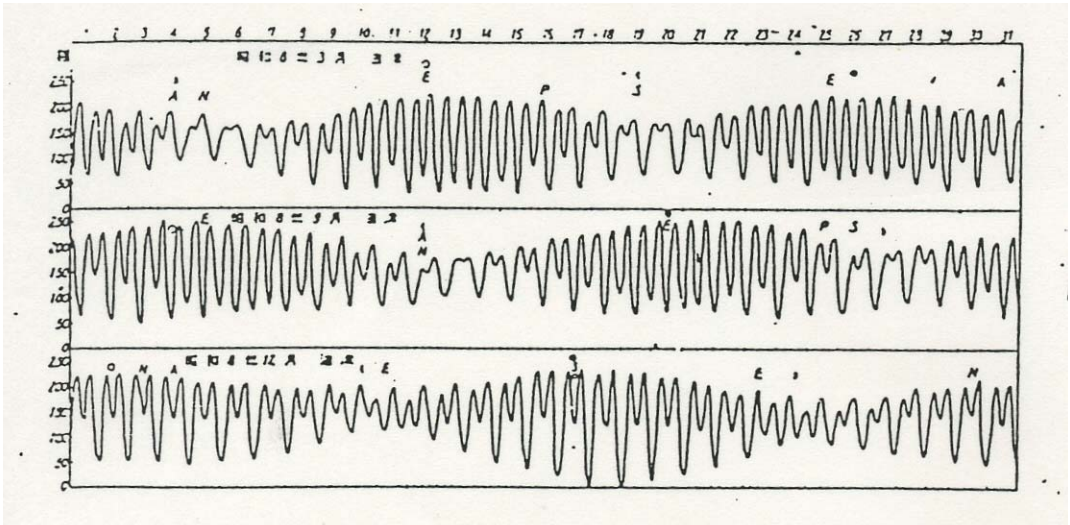


Fig 9 Actual tidal curve at Aburatsu Port, Kagoshima Prefecture, in March 1933

Ebbs and floods occur twice a day, but the heights and depths are not always the same. This is because the axis of the rotation of the earth is inclined to the polar star, or the north pole.

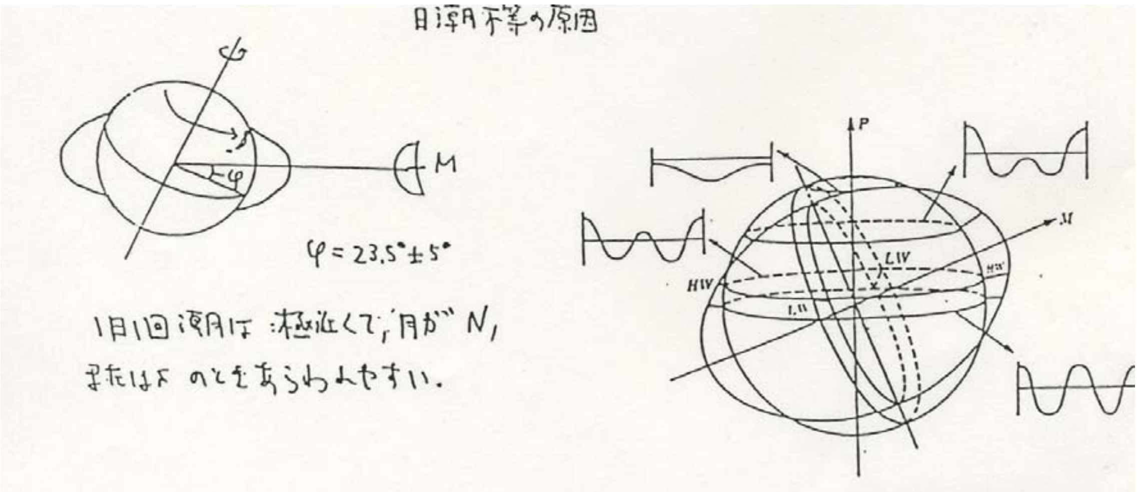


Fig. 10　Why uneven ebbs-floods occur

［Problem］　Uneven ebbs-floods occur larger at a place apart more from the equator

Hint: See Fig. 10

In a sea region of high latitude, tidal pattern of one daily ebb-and-flood is likely to occur.

[Problem] Why is the tidal pattern of a daily ebb-and-flood likely to occur in a sea at high latitude?

11.7　Principal Components of Astronomical Tide

As shown in the predicted tidal curve at Aburatsu Port in Fig. 8, the astronomical tidal curve takes a curve of “beating waves,” which is expressed by the sum of four component of harmonic functions (=sinusoidal functions). We call them the “Four Principal Components of Astronomical Tide.”

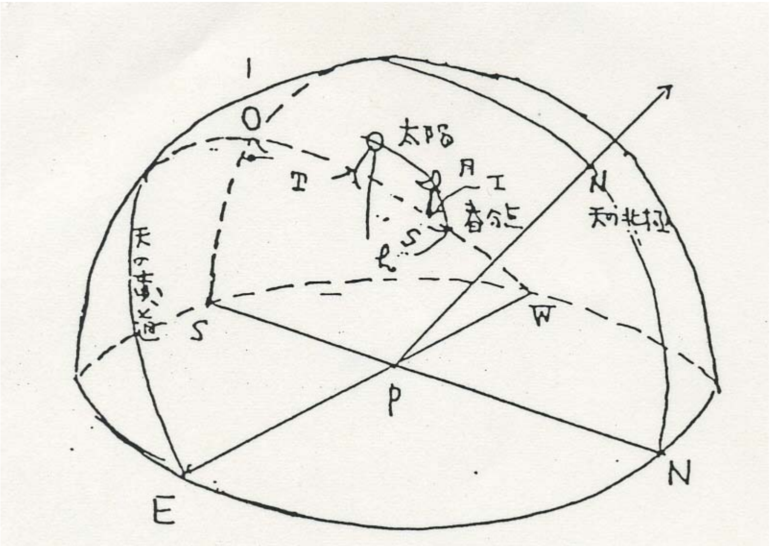


Fig. 11 Celestial Sphere and hour angle

If we observe the sky with a geocentric view, it has a hemispherical shape covering the ground; we call this hemisphere the "celestial sphere". The sun, the moon, and stars appear to revolve around the polar star once a day.

We call the great circle at 90 degrees from the polar star "the equator," similar to that of the Earth. We call the great circle passing the north pole and the southernmost point "the culmination line." The sun crosses the culmination line at noon every day, and moves westward approximately 15 degrees/h. We call the angle from the culmination line to west the "the hour angle;" it is measured not only in "degrees" but also in "hour and minutes."

Let the hour angle of the sun observed at the present place be T. Let the solar longitude of the sun be h, that of the moon be s, and the factors converting from the solar to celestial longitudes of the sun and the moon as ν and ξ; the hour angle of the moon is then given by $T+(h-s)+(\xi-\nu)$. Hence, the most prominent component of the tidal component M_2 is given by

$$M_2 = F_{M2}\cos(2T+2h-2s+2\xi-2\nu) \qquad (20)$$

The next largest component is that of the sun and is given by

$$S_2 = F_{S2}\cos(2T) \qquad (21)$$

The unevenness of the two ebbs-floods can be expressed by adding the next two components

$$O_1 = F_{O1}\cos(T+h-2s+2\xi-\nu+\pi/2) \qquad (22)$$

and

$$K_1 = F_{K1}\cos(T + h + \pi/2) \quad (23)$$

We call these four components as "Four principal tide components." The suffix "1" and "2" shows diurnal and bi-diurnal components, respectively.

If we consider that the orbit of the moon is not a circle but an ellipsoid, and the orbit itself changes in a half period, we should introduce more tidal components in addition to the four principal components, for example, $N_2, P_2, Q_1, L_2 \cdots$.

Theoretically more than 40 components can be considered; however, in general, it is sufficient to predict the astronomical tide using 14 components. In addition, we can consider the tidal component induced by the meteorological change "meteorological tide," which changes over a period of one year and is written in the symbols Sa.

11.9 Roche's limit

The formula (16-a) shows that the amount of the acceleration induced by the tidal effect is only 10^{-7} times (ten million per one) of the gravity g. In (16-a), R is the radius of the Earth, M is the mass of the moon, and g is the gravitational acceleration of the earth. We next consider the tidal acceleration on the moon surface. We should use R as the radius of the moon (1/4 of that of the Earth); M, the mass of the Earth (20 times that of the moon); and g, the acceleration on the moon's surface (g_M = 1/6 that on the Earth's surface). Then (16-a) for moon is given by (24).

$$\vec{F}_v = 1.0\times10^{-4} g_M \quad (24)$$

It is still at odds of ten thousand to one. However, if we assume that the distance between the earth and the moon, D, becomes shorter by 1/22, then what will happen? We have (25), which means that the gravitational acceleration of the moon is equal to the tidal acceleration!! If this were true, stones on the moon's surface will float in space, and the moon will not exist.

$$\vec{F}_v = 1.0 g_M \quad (25)$$

We call this limit "Roche's limit," and if this limit is exceeded, the moon will be broken leading to rings like those in Saturn Ring.

[Problem] If we consider only the third component, the limit is given by $D_R = \sqrt[3]{2R\rho_1/(R_2\rho_2)}\times R$.

[Problem] Without using the Taylor series approximation, estimate the limit more accurately by using the original formulae.

11.10 Dynamic Tidal Theory

In the previous section, we discussed in detail the static tidal theory, in which the ocean surface takes the form of an iso-potential surface. We consider a simple pendulum whose upper end we can pull with our fingers. The eigenvalue period is assumed to be P_0, and we oscillate it with a period P. Then

(a) if $P < P_0$, the pendulum moves in the same phase as the fingers, but

(b) if $P > P_0$, the pendulum moves in an inverse phase with the fingers

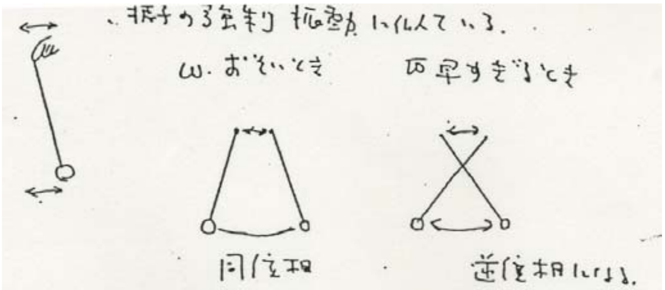


Fig. 12 Pendulum of forced oscillation

An identical phenomenon takes place in tides. We assume that the length of the equator is 40,000 km, and there is a ditch with a depth of 4,000 meters along the equator. Since the wave speed is given by $V = \sqrt{gD}$, $V = 198.0$ m/s = 713 km/h. It takes 56 h for the tidal wave to run around the equator once. The tidal wave cannot reach the speed of rotation of the Earth.

The water surface solution for this compulsive case is given by (26) and (27)

$$\eta_0 = \frac{3}{4}\frac{GMR^2}{D^3}\cos 2(\omega t + \varepsilon) \qquad (26)$$

Solution for forced oscillation is given by、

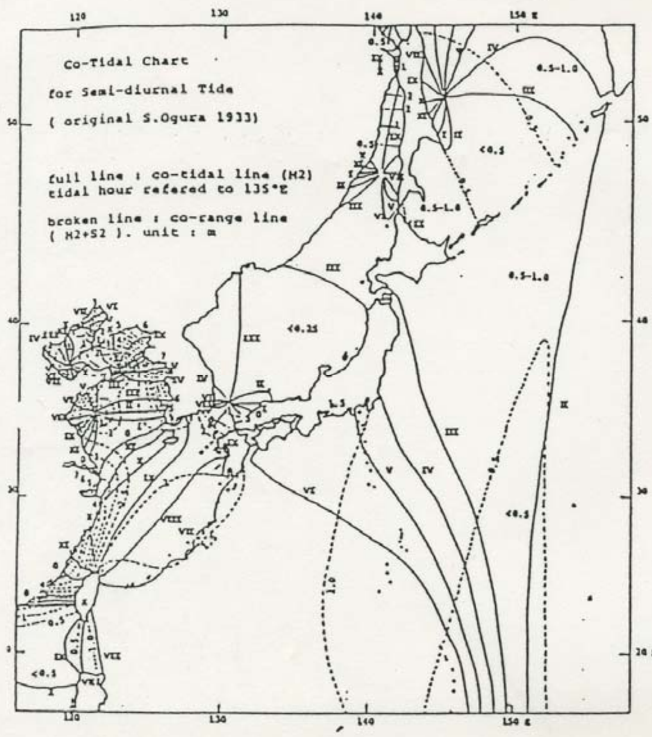
$$\eta = \frac{gD}{gD - \omega^2 R^2}\eta_0 \qquad (26)$$

where, $\omega = 2\pi/P$.

The case that $\sqrt{gD} < \omega R$ is a case of a forced oscillation, and η changes out phase with η_0.

11.11 Ridge line of the tidal waves and Kelvin waves

Isochronal line of a ridge of the M_2 tide around the Japanese Islands and their vicinities is shown in Fig 13.



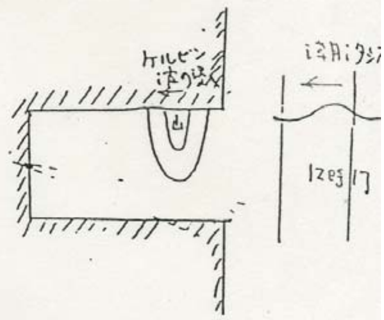
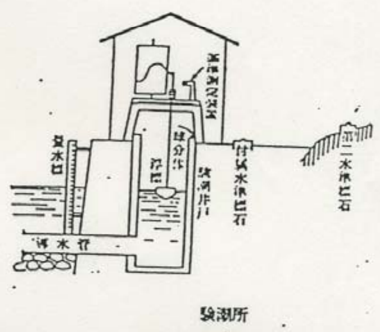



Fig 13. Ridge lines of M2 component Fig. 14 Kelvin's wave Fig. 15 Tidal station

Note that three non-tidal points (nodal points) appear in Bo Hai Bay, Yellow Sea, and in Tsushima Straights. Tidal waves are a ocean long wave strongly influenced by the rotation of the Earth. The governing equations are given as follows;

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{28}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{29}$$

where f is the coefficient of Coriolis force derived fron the rotation of the Earth. Vertically integrated equation of mass conservation is given by

$$\frac{\partial \eta}{\partial t} = -D\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{30}$$

We assume that the solution has the style of $v = 0, u = e^{i(kx-\sigma t)}, \eta = Y(y)e^{i(kx-\sigma t)}$, then we can obtain the solution satisfying (28)-(30) as follows;

$$\eta = He^{-f/cy}e^{i(kx-\sigma t)} \tag{31}$$

$$u = \sqrt{g/D}\eta \tag{32}$$

where $c = \pm\sqrt{gD}$. We can recognize that the tidal waves are stationary waves composed of overlapped Kelvin wave components in marginal seas. A non-tidal point is nodal point of the stationary waves.

[Problem] Prove (31) and (32) satisfies equations (28) to (30).

[Problem] If the shape of Yellow Sea is a rectangle, the nodal points will be arranged in even interval on the central axis line in the case that no bottom friction acts.

[Problem] How are nodal points arranged in the case that bottom friction acts? Along which side coast the amplitude of tidal waves becomes large, along the eastside coast or west side coast?

[Problem] In which direction do isochronal lines of tidal waves round around a nodal point, clockwise or counterclockwise in the northern hemisphere? How it will be in the southern hemisphere?